The diagonal Ising susceptibility

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Abstract. We use the recently derived form factor expansions of the diagonal two-point correlation function of the square Ising model to study the susceptibility for a magnetic field applied only to one diagonal of the lattice, for the isotropic Ising model. We exactly evaluate the one and two particle contributions $\chi_d^{(1)}$ and $\chi_d^{(2)}$ of the corresponding susceptibility, and obtain linear differential equations for the three and four particle contributions, as well as the five particle contribution $\chi_d^{(5)}(t)$, but only modulo a given prime. We use these exact linear differential equations to show that, not only the russian-doll structure, but also the direct sum structure on the linear differential operators for the n-particle contributions $\chi_d^{(n)}$ are quite directly inherited from the direct sum structure on the form factors $f^{(n)}$. We show that the n^{th} particle contributions $\chi_d^{(n)}$ have their singularities at roots of unity. These singularities become dense on the unit circle $|\sinh 2E_v/kT \sinh 2E_h/kT| = 1$ as $n \rightarrow \infty$.

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1. Introduction

The magnetic susceptibility χ of the two dimensional Ising model is expressed in terms of the two point correlation function $C(M,N) = \langle \sigma_{0,0}\sigma_{M,N}\rangle$ as

$$kT \cdot \chi = \sum_{M=-\infty}^{\infty} \sum_{N=-\infty}^{\infty} (C(M,N) - \mathcal{M}(0)^2) \qquad (1)$$

where $\mathcal{M}(0)$ is the spontaneous magnetization (which vanishes for $T > T_c$). This susceptibility has been studied for over three decades by use of the form factor representations of the correlation function [1]-[5]

$$C(M,N)_{\pm} \;=\; (1-t)^{1/4} \cdot \sum_n C^{(n)}(M,N) \qquad (2)$$

where the subscript $+(-)$ denotes $T > T_c$ $(T < T_c)$. For $T > T_c$ the variable t is $t = (\sinh 2E_v/kT \sinh 2E_h/kT)^2$ and for $T < T_c$ is $t = (\sinh 2E_v/kT \sinh 2E_h/kT)^{-2}$ where E_v and E_h are the vertical and horizontal interaction constants. The sum is over odd (even) values of the integer n for $T > T_c$ $(T < T_c)$ and the $C^{(n)}(M,N)$ are explicit n fold integrals. Using this form factor decomposition (2) in the expression for the susceptibility (1) we find

$$kT \cdot \chi_{\pm} \;=\; (1-t)^{1/4} \cdot \sum_n \tilde{\chi}^{(n)} \qquad (3)$$

where the $\tilde{\chi}^{(n)}$ can be expressed in terms of double sum of the form factors:

$$\tilde{\chi}^{(n)} \;=\; \sum_{N=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} C^{(n)}(M,N) \qquad (4)$$

The study of these $\tilde{\chi}^{(n)}$ was initiated in [1] with the explicit evaluation of the integrals for $\tilde{\chi}^{(1)}$ and $\tilde{\chi}^{(2)}$. However, a real understanding of the analytic structure of $\tilde{\chi}^{(n)}$ began only in 1999 and 2000, with the demonstration by Nickel [4, 5], for the isotropic Ising model ($E_v = E_h = E$), that $\tilde{\chi}^{(n)}$ has a set of singularities, lying on the unit circle $|\sinh 2E/kT| = 1$, which become dense in the limit $n \to \infty$. If these singularities do not cancel in the full sum (3) then the susceptibility will have a *natural boundary* at $|\sinh 2E/kT| = 1$.

The existence of a natural boundary in the susceptibility is a profound effect not envisioned in the traditional scaling, and renormalization, description of critical phenomena. In order to obtain further insight into existence of such natural boundary, several of the present authors have made a detailed study of $\tilde{\chi}^{(3)}$ and $\tilde{\chi}^{(4)}$ in [6]-[9].

The integrals for $\tilde{\chi}^{(n)}$, as explicitly written out in [4, 5], are quite complicated, and, for that reason, it is difficult to extend the analysis of [6]-[9] to $\tilde{\chi}^{(n)}$ with $n \geq 5$. A direct attack [10] on $\tilde{\chi}^{(5)}$ indicates that more than 6000 terms and, for $\tilde{\chi}^{(6)}$, probably more than 20000 terms in the power series expansion in t are needed in order to find the linear differential equations which they satisfy. Therefore it would be most useful to study "model integrals" that are simpler to analyze, and which incorporate significant features of $\tilde{\chi}^{(n)}$. Several such "model integrals" have been previously studied [11] and revealed a rich structure of singularities beyond those found by Nickel [4, 5].

In this paper we introduce what is probably the most physical simplification of the Ising susceptibility which retains the property of having singularities on the unit circle $|t| = 1$. This model is obtained by considering the isotropic Ising model with a magnetic field *which acts only on one diagonal* of the lattice. The magnetic susceptibility for this diagonal field will be, then, given by the diagonal analogue of (1):

$$kT \cdot \chi_d \;=\; \sum_{N=-\infty}^{\infty} \left(C(N,N) \;-\mathcal{M}(0)^2\right). \qquad (5)$$

This expression¶ should be much easier to study than the full susceptibility. This stems from the form factor decomposition of the diagonal two-point correlations $C(N,N)$, that has been, recently, presented [12] and proven [13], and which is much simpler than the decomposition obtained directly from [1]. In particular for $T < T_c$

$$C(N,N)_- \; = \; (1-t)^{1/4} \cdot \Big(1 + \sum_{n=1}^{\infty} f^{(2n)}(N,t)\Big) \qquad (6)$$

with

$$\begin{aligned} f^{(2n)}(N,t) = \; & \frac{t^{n(n+|N|)}}{(n!)^2} \, \frac{1}{\pi^{2n}} \int_0^1 \cdots \int_0^1 \prod_{k=1}^{2n} dx_k \cdot x_k^{|N|} \\ & \times \prod_{j=1}^{n} \Big(\frac{x_{2j-1}(1-x_{2j})(1-tx_{2j})}{x_{2j}(1-x_{2j-1})(1-t\, x_{2j-1})} \Big)^{1/2} \\ & \times \prod_{1 \le j \le n} \, \prod_{1 \le k \le n} (1 - t\, x_{2j-1}\, x_{2k})^{-2} \qquad (7) \\ & \times \prod_{1 \le j < k \le n} (x_{2j-1} - x_{2k-1})^2 \, (x_{2j} - x_{2k})^2 \end{aligned}$$

and for $T > T_c$

$$C(N,N)_+ \; = \; (1-t)^{1/4} \cdot \sum_{n=0}^{\infty} f^{(2n+1)}(N,t) \qquad (8)$$

with

$$\begin{aligned} f^{(2n+1)}(N,t) \; = \; & \frac{t^{n(n+1)+|N|(n+1/2)}}{\pi^{2n+1} n!(n+1)!} \cdot \int_0^1 \cdots \int_0^1 \prod_{k=1}^{2n+1} dx_k x_k^{|N|} \\ & \times \prod_{j=1}^{n} \Big((1-x_{2j})(1-t\, x_{2j}) \cdot x_{2j} \Big)^{1/2} \\ & \times \prod_{j=1}^{n+1} \Big((1-x_{2j-1})(1-t\, x_{2j-1}) \cdot x_{2j-1} \Big)^{-1/2} \\ & \times \prod_{1 \le j \le n+1} \, \prod_{1 \le k \le n} (1 - t\, x_{2j-1}\, x_{2k})^{-2} \qquad (9) \\ & \times \prod_{1 \le j < k \le n+1} (x_{2j-1} - x_{2k-1})^2 \cdot \prod_{1 \le j < k \le n} (x_{2j} - x_{2k})^2 \end{aligned}$$

From now on, the integer N should be understood as $|N|$ when evaluated.

Thus, if we use (6) and (8) in (5), and evaluate the sum on N as a geometric series, we obtain for $T < T_c$

$$kT \cdot \chi_{d-}(t) \; = \; (1-t)^{1/4} \cdot \sum_{n=1}^{\infty} \tilde{\chi}_d^{(2n)}(t) \qquad (10)$$

with

$$\tilde{\chi}_d^{(2n)}(t) \; = \; \frac{t^{n^2}}{(n!)^2} \, \frac{1}{\pi^{2n}} \cdot \int_0^1 \cdots \int_0^1 \prod_{k=1}^{2n} dx_k \cdot \frac{1 + t^n \, x_1 \cdots x_{2n}}{1 - t^n \, x_1 \cdots x_{2n}}$$

¶ Such partial sums have already been introduced and the asymptotic expansion of their short term part considered in Orrick *et al.* [14].

$$\times \prod_{j=1}^{n} \Big({{ x_{2j-1} (1-x_{2j}) (1-tx_{2j})} \over { x_{2j} (1-x_{2j-1}) (1 \, - t\, x_{2j-1})}} \Big)^{1/2}$$
$$\times \prod_{1 \le j \le n} \prod_{1 \le k \le n} (1 \, - t\, x_{2j-1}\, x_{2k})^{-2}$$
$$\times \prod_{1 \le j < k \le n} (x_{2j-1} - x_{2k-1})^2 \, (x_{2j} - x_{2k})^2 \qquad (11)$$

and for $T > T_c$

$$kT \cdot \chi_{d+}(t) \; = \; (1-t)^{1/4} \cdot \sum_{n=0}^{\infty} \tilde{\chi}^{(2n+1)}_d(t) \qquad (12)$$

with

$$\tilde{\chi}^{(2n+1)}_d(t) \; = \; {{ t^{n(n+1))}} \over { \pi^{2n+1} n! \, (n+1)!}} \cdot \int_0^1 \cdots \int_0^1 \prod_{k=1}^{2n+1} dx_k$$
$$\times {{ 1 \, + t^{n+1/2} \, x_1 \cdots x_{2n+1}} \over { 1 \, - t^{n+1/2} \, x_1 \cdots x_{2n+1} }} \cdot \prod_{j=1}^{n} \Big((1-x_{2j})(1 \, - t\, x_{2j}) \cdot x_{2j} \Big)^{1/2}$$
$$\times \prod_{j=1}^{n+1} \Big((1 \, - x_{2j-1})(1 \, - t\, x_{2j-1}) \cdot x_{2j-1} \Big)^{-1/2}$$
$$\times \prod_{1 \le j \le n+1} \prod_{1 \le k \le n} (1 \, - t\, x_{2j-1}\, x_{2k})^{-2} \qquad (13)$$
$$\times \prod_{1 \le j < k \le n+1} (x_{2j-1} - x_{2k-1})^2 \prod_{1 \le j < k \le n} (x_{2j} - x_{2k})^2.$$

The expressions (11) and (13) are, indeed, much simpler than the corresponding expressions for $\tilde{\chi}^{(n)}$ given in [1]-[9].

In this paper we analyze the contributions $\tilde{\chi}^{(n)}_d(t)$ to the diagonal susceptibility. In section (2) we analytically evaluate the integrals for $n = 1$ and 2. In section (3) we present, and analyze, the ordinary linear differential equations satisfied by $\tilde{\chi}^{(3)}_d(t)$ and $\tilde{\chi}^{(4)}_d(t)$, as well as $\tilde{\chi}^{(5)}_d(t)$ modulo a given prime. These linear differential equations have a direct sum decomposition which we relate in section (5) to the direct sum decomposition found in [13] for the form factors $f^{(n)}(N, t)$. In section (4) we extract the singularities in the integral representations (11) and (13) for $\tilde{\chi}^{(n)}_d(t)$. We find that all the singularities of $\tilde{\chi}^{(2n)}_d(t)$ are at the roots of unity $t^n \, = 1$, while, for $\tilde{\chi}^{(2n+1)}_d(t)$, they are at $t^{n+1/2} \, = 1$. These singularities for $t \ne 1$ are the counterparts, for $\tilde{\chi}^{(n)}_d(t)$, of the singularities on the unit circle found by Nickel [4, 5] for the full susceptibility $\tilde{\chi}$. However, unlike $\tilde{\chi}^{(n)}(t)$ which was shown in [6]-[11] to have many other singularities which lie outside the unit circle $|\sinh 2E/kT| = 1$, the diagonal $\tilde{\chi}^{(n)}_d(t)$ has no further singularities other than those which satisfy $t^n \, = 1$ or $t^{n+1/2} \, = 1$. We conclude in section (6) with a discussion of the several different types of singularities in $\chi_d(t)$ which follow from the singularities in $\tilde{\chi}^{(n)}_d(t)$.

2. Evaluation of $\tilde{\chi}_d^{(1)}(t)$ and $\tilde{\chi}_d^{(2)}(t)$

The contribution of $\tilde{\chi}_d^{(1)}(t)$ is explicitly given from (13) as

$$\tilde{\chi}_d^{(1)}(t) \;=\; {{1} \over {\pi}} \int_0^1 dx \cdot {{1 \, + t^{1/2} \, x} \over {1 \, - t^{1/2} \, x}} \cdot [(1-x)(1 \, - t\, x) \, x]^{-1/2} \qquad (14)$$

which, setting $x \, = \, \sin^2\theta$ is written as

$$\tilde{\chi}_d^{(1)}(t) \;=\; {{2} \over {\pi}} \int_0^{\pi/2} d\theta \cdot {{1 \, + t^{1/2} \sin^2\theta} \over {1 \, - t^{1/2} \sin^2\theta}} \cdot (1 \, - t \sin^2\theta)^{-1/2}. \qquad (15)$$

Recalling the definitions of the elliptic integrals of the first kind

$$K(t) \;=\; \int_0^{\pi/2} {{d\theta} \over {(1 \, - t \, \sin^2\theta)^{1/2}}} \qquad (16)$$

and of the third kind

$$\Pi_1(\nu, t) \;=\; \int_0^{\pi/2} {{d\theta} \over {(1 \, + \nu \sin^2\theta)(1 \, - t \, \sin^2\theta)^{1/2}}} \qquad (17)$$

we may write (15) as

$$\tilde{\chi}_d^{(1)}(t) \;=\; {{2} \over {\pi}} \cdot \Bigl(2 \, \Pi_1(-t^{1/2}, \, t) \;\; - K(t) \Bigr). \qquad (18)$$

Thus, if we use the identity [15]

$$\begin{aligned} {{2} \over {\pi}} \cdot \Bigl(\Pi_1(\nu, \, t) \; + \Pi_1(t/\nu, \, t) \Bigr) \;\; &= \qquad (19) \\ &{{2} \over {\pi}} \cdot K(t) \;\; + \; [(1+\nu)(1+t/\nu)]^{-1/2} \end{aligned}$$

with $\nu \, = \, -t^{1/2}$ we find:

$$\tilde{\chi}_d^{(1)}(t) \;=\; {{1} \over {1 \; - t^{1/2}}}. \qquad (20)$$

It is instructive to derive (20) without recourse to identities on elliptic integrals. We first rewrite (14) as a contour integral

$$\tilde{\chi}_d^{(1)}(t) \;=\; {{1} \over {2\pi i}} \oint {{dz} \over {z}} \cdot {{1 \, + z} \over {1 - z}} \cdot [(1 - t^{1/2} z)(1 - t^{1/2} z^{-1})]^{-1/2} \qquad (21)$$

on the contour $|z| < 1$ (which is, in fact, the form in which the form factors $f^{(n)}(N, t)$ were originally derived [13]). The integrand in (21) is antisymmetric if we send $z \rightarrow 1/z$. Therefore

$$\tilde{\chi}_d^{(1)}(t) \;=\; -\tilde{\chi}_d^{(1)}(t) \;\; - (\text{residue at } \; z = 1) \qquad (22)$$

The residue at $z = 1$ is easily evaluated as

$$(\text{residue at } \; z = 1) \;\; = \;\; - {{2} \over {1 - t^{1/2}}} \qquad (23)$$

and thus using (23) in (22) we again obtain the result (20).

To compute $\tilde{\chi}_d^{(2)}(t)$ we also use the contour integral method, and rewrite (11) with $n = 1$ as

$$\begin{aligned} \tilde{\chi}_d^{(2)}(t) \;=\; &{{1} \over {(2\pi i)^2}} \oint dz_1 \oint dz_2 \qquad (24) \\ &\times {{1 \, + z_1 z_2} \over {(1 \, - z_1 z_2)^3}} \cdot \left[{{(1 \, - t^{1/2} z_2)(1 \, - t^{1/2} z_2^{-1})} \over {(1 \, - t^{1/2} z_1)(1 \, - t^{1/2} z_1^{-1})}} \right]^{1/2} \end{aligned}$$

on the contour $|z_k| < 1$ for $k = 1, 2$. As was the case for $\tilde{\chi}_d^{(1)}(t)$, we note that the integrand of (24) is antisymmetric if we send $z_1 \to 1/z_1,\ z_2 \to 1/z_2$. Therefore

$$\tilde{\chi}_d^{(2)}(t) \;=\; -\frac{\pi i}{(2\pi i)^2} \oint dz_1 (\text{residue at } \; z_2 = z_1^{-1}) \tag{25}$$

where:

$$\begin{aligned}(\text{residue at } \; z_2 = z_1^{-1}) \;=\; & z_1^{-3}[(1 - t^{1/2} z_1)(1 - t^{1/2} z_1^{-1}]^{-1/2} \\ & \times \frac{1}{2} \frac{d^2}{dz_2^2} \{(1 + z_1 z_2)[(1 - t^{1/2} z_2)(1 - t^{1/2} z_2^{-1}]^{1/2}\}|_{z_2 = z_1^{-1}}.\end{aligned} \tag{26}$$

When (26) is evaluated, and substituted into (25), the resulting integral over z_1 has only poles. Keeping only those terms which give nonvanishing contributions, we find

$$\tilde{\chi}_d^{(2)}(t) \;=\; \frac{1}{8\pi i} \oint dz_1 \frac{t}{(1 - t^{1/2} z_1)(z_1 - t^{1/2})} \;=\; \frac{t}{4\,(1-t)}. \tag{27}$$

It should be noted that neither $\tilde{\chi}_d^{(1)}(t)$, nor $\tilde{\chi}_d^{(2)}(t)$, contain logarithms even though there are logarithms in both $f^{(1)}(N, t)$ and $f^{(2)}(N, t)$ for all N. This is to be contrasted with the corresponding results for the full susceptibility where it was seen in [1] that $\tilde{\chi}^{(n)}(t)$ has no logarithms for $n = 1$, but does have a term in $\ln t$ for $n = 2$.

3. Linear differential equations for $\tilde{\chi}_d^{(3)}(t)$, $\tilde{\chi}_d^{(4)}(t)$ and $\tilde{\chi}_d^{(5)}(t)$

We now turn to $\tilde{\chi}_d^{(n)}(t)$ for $n \ge 3$. When written in contour integral form the integrands still have the property of being antisymmetric when $z_k \to 1/z_k$. Unfortunately this property is not sufficient to reduce the computation to integrals that can all be evaluated by residues. Therefore we do not have an explicit evaluation in terms of elementary functions, and we continue our study by using formal computer computations to determine the linear differential equations satisfied by $\tilde{\chi}_d^{(3)}(t)$, $\tilde{\chi}_d^{(4)}(t)$ and $\tilde{\chi}_d^{(5)}(t)$, as was done for $\tilde{\chi}^{(3)}$ and $\tilde{\chi}^{(4)}$ for the full susceptibility in [6]-[9] and for the $f^{(n)}(N, t)$ in [12]. We present the results of these computer calculations for $\tilde{\chi}_d^{(3)}(t)$, $\tilde{\chi}_d^{(4)}(t)$ and $\tilde{\chi}_d^{(5)}(t)$ separately.

3.1. Linear differential equation for $\tilde{\chi}_d^{(3)}(t)$

For $\chi_d^{(3)}$ we chose $x = t^{1/2} = \sinh 2E_v/kT \, \sinh 2E_h/kT$ as our independent variable. We find that the linear differential operator for $\tilde{\chi}_d^{(3)}(x)$ is of order six, and has the direct sum decomposition

$$\mathcal{L}_6^{(3)} \;=\; L_1^{(3)} \oplus L_2^{(3)} \oplus L_3^{(3)} \tag{28}$$

with

$$L_1^{(3)} \;=\; Dx \,+\, \frac{1}{x-1}, \tag{29}$$

$$L_2^{(3)} \;=\; Dx^2 \;+ 2\,\frac{(1+2\,x)}{(1+x)\,(x-1)} \cdot Dx \,+\, \frac{1+2\,x}{(1+x)\,(x-1)\,x}, \tag{30}$$

$$
\begin{aligned}
L_3^{(3)} = \; & Dx^3 \\
& + \frac{3}{2} \, \frac{\left(8\,x^6 + 36\,x^5 + 63\,x^4 + 62\,x^3 + 21\,x^2 - 6\,x - 4\right)}{(x+2)\,(1+2\,x)\,(1+x)\,(x-1)\,(1+x+x^2)\,x} \cdot Dx^2 \\
& + \frac{n_1}{(x+2)\,(1+2\,x)\,(1+x)^2\,(x-1)^2\,(1+x+x^2)\,x^2} \cdot Dx \\
& + \frac{n_0}{(x+2)\,(1+2\,x)\,(x-1)^3\,(1+x+x^2)\,(1+x)^2 x^2}
\end{aligned} \qquad (31)
$$

with:

$$
\begin{aligned}
n_0 \; &= \; 2\,x^8 + 8\,x^7 - 7\,x^6 - 13\,x^5 - 58\,x^4 - 88\,x^3 - 52\,x^2 - 13\,x + 5, \\
n_1 \; &= \; 14\,x^8 + 71\,x^7 + 146\,x^6 + 170\,x^5 + 38\,x^4 \\
& \qquad - 112\,x^3 - 94\,x^2 - 19\,x + 2.
\end{aligned}
$$

The solution of $L_1^{(3)}$ is in fact (up to a constant) $\tilde{\chi}_d^{(1)}(t)$ as given in (20). This means that $\tilde{\chi}_d^{(1)}(t)$ is actually solution of the Fuchsian linear differential operator $\mathcal{L}_6^{(3)}$ corresponding to $\tilde{\chi}_d^{(3)}(t)$. We recover then the "russian-doll structure" noticed in [8, 9] for the third contribution $\tilde{\chi}^{(3)}$ to the full susceptibility.

The linear differential operator of order two, $L_2^{(3)}$ has, at $x = 0, 1, -1$ and $x = \infty$, respectively the exponents $(0,1), (-2,0), (0,0)$, and $(1,2)$. Some manipulations on the formal solutions, at $x = 0$, give the result that the solution analytic at $x = 0$ reads

$$
sol(L_2^{(3)}) \; = \; \frac{1}{x-1} \cdot K(x^2) \; + \; \frac{1}{(x-1)^2} \cdot E(x^2) \qquad (32)
$$

with:

$$
K(y) = \, F\,(1/2, 1/2; 1; y)\,, \qquad E(y) = \, F\,(1/2, -1/2; 1; y) \qquad (33)
$$

(where we have abused conventional notation by omitting the factor of $\pi/2$ in the canonical definition‡ of the complete elliptic integrals K and E).

We have not found the explicit solution of the linear differential operator $L_3^{(3)}$ which has the following regular singular points and exponents:

$$
\begin{array}{lll}
1 + x + x^2 = 0, & \rho = 0, 1, 7/2 \; \rightarrow & x^{7/2}, \\
x = \; 0 & \rho = 0, 0, 0 \quad \rightarrow & log^2 \;\; terms, \\
x = \; 1 & \rho = -2, -1, 1 \rightarrow & x^{-2}, \; x^{-1}, \\
x = -1 & \rho = 0, 0, 0 \quad \rightarrow & log^2 \;\; terms, \\
x = \; \infty & \rho = 1, 1, 1 \quad \rightarrow & log^2 \;\; terms.
\end{array} \qquad (34)
$$

The singularities at $x = 2, -1/2$ are apparent.

3.2. Linear differential equation for $\tilde{\chi}_d^{(4)}(t)$

For $\tilde{\chi}_d^{(4)}(t)$ we use t as the independent variable. The linear differential operator for $\tilde{\chi}_d^{(4)}(t)$ is of order eight, and has the direct sum decomposition

$$
\mathcal{L}_8^{(4)} \; = \; L_1^{(4)} \oplus L_3^{(4)} \oplus L_4^{(4)} \qquad (35)
$$

‡ In maple's notations, for $t = k^2$ (k is the modulus), $K(t) = K(k^2)$ in (33) reads : $hypergeom([1/2, 1/2], [1], t) = 2/\pi \cdot EllipticK(k)$, but reads $2/\pi \cdot EllipticK[k^2]$ in Mathematica.

with

$$L_1^{(4)} \; = \; Dt \, + \, \frac{1}{(t-1)\, t}, \tag{36}$$

$$L_3^{(4)} \; = \; Dt^3 \, + \, \frac{\left(5\, t^2 + 6\, t - 1\right)}{(1+t)\,\, (t-1)\, t} \cdot Dt^2 \, + \, \frac{\left(3\, t^3 + 6\, t^2 - 2\, t - 1\right)}{(1+t)\, t^2\, (t-1)^2} \cdot Dt \\ - \frac{3}{2\,(1+t)\,(t-1)\, t^2}, \tag{37}$$

$$L_4^{(4)} \; = \; Dt^4 \, + \, \frac{\left(7\, t^4 - 68\, t^3 - 114\, t^2 + 52\, t - 5\right)}{(t+1)\,(t^2 - 10\, t + 1)\,(t-1)\, t} \cdot Dt^3 \\ + 2\, \frac{\left(5\, t^5 - 55\, t^4 - 169\, t^3 + 149\, t^2 - 28\, t + 2\right)}{(t+1)\,(t^2 - 10\, t + 1)\, t^2\,(t-1)^2} \cdot Dt^2 \\ + 2\, \frac{\left(t^4 - 13\, t^3 - 129\, t^2 + 49\, t - 4\right)}{(t+1)\,(t^2 - 10\, t + 1)\, t^2\,(t-1)^2} \cdot Dt \\ - 3\, \frac{(t+1)^2}{(t^2 - 10\, t + 1)\,(t-1)^2\, t^3}. \tag{38}$$

The solution of $L_1^{(4)}$ is (up to a constant) the function $\tilde{\chi}_d^{(2)}(t)$ given in (27). Here also, $\tilde{\chi}_d^{(2)}(t)$ is solution of the Fuchsian linear differential operator $\mathcal{L}_8^{(4)}$ corresponding to $\tilde{\chi}_d^{(4)}(t)$, and we recover, again, the russian-doll structure noticed in [8, 9] for the fourth contribution $\tilde{\chi}^{(4)}$ to the full susceptibility.

The solution of $L_3^{(4)}$ is found to be (with notations (33)):

$$sol(L_3^{(4)}) \; = \; -K(t)^2 \, + \, \frac{1+t}{(1-t)^2} \cdot E(t)^2 \, + \, \frac{2t}{t-1} \cdot K(t)\, E(t). \tag{39}$$

We have not found the explicit solution for the operator $L_4^{(4)}$ which has the following regular singular points and exponents

$$\begin{array}{llll} t = \;\; 0, & \rho = 0,0,0,1 & \rightarrow & log^3 \quad terms, \\ t = \;\; 1, & \rho = -2,-1,0,1 & \rightarrow & t^{-2},\; t^{-1},\; log \quad term, \\ t = -1, & \rho = 0,1,2,7 & \rightarrow & t^7\, log \quad term, \\ t = \;\; \infty, & \rho = 0,0,0,1 & \rightarrow & log^3 \quad terms. \end{array} \tag{40}$$

The singularities at the roots of $t^2 - 10t + 1 = 0$ are apparent.

3.3. Linear differential equation for $\tilde{\chi}_d^{(5)}(t)$

For $\tilde{\chi}_d^{(5)}$ we chose, again, $x = t^{1/2} = \sinh 2E_v/kT \, \sinh 2E_h/kT$ as our independent variable. The first terms of the series expansion of $\tilde{\chi}_d^{(5)}(x)$ read :

$$\tilde{\chi}_d^{(5)}(x) \; = \; \frac{3}{262144} \cdot x^{12} \, + \, \frac{39}{1048576} \cdot x^{14} \, + \, \frac{5085}{67108864} \cdot x^{16} \\ + \, \frac{9}{67108864} \cdot x^{17} \, + \, \frac{33405}{268435456} \cdot x^{18} \, + \, \frac{315}{536870912} \cdot x^{19} \, + \cdots \tag{41}$$

Now, the calculations, in order to get the linear differential operator for $\tilde{\chi}_d^{(5)}(x)$, become really huge. For that reason, we introduce a "modular" strategy which

amounts to generating large series *modulo a given prime*, and then deduce, from a Padé-Hermite procedure, the linear differential operator for $\tilde{\chi}_d^{(5)}(x)$ *modulo that prime*. We have generated 3000 coefficients for the series expansion of $\tilde{\chi}_d^{(5)}(x)$ modulo a given prime (here 32003), and actually found a linear differential equation *modulo that prime* of order 25, 26, $\cdots$, using 2200 terms in the series expansion. One can also obtain linear differential equations of smaller order for $\tilde{\chi}_d^{(5)}(x)$, but where *more terms* (2500, 2600, 2800, $\cdots$), are needed. The polynomial corresponding to apparent singularities in front of the highest derivative is now very large. The smallest order one can reach is 19, and the linear differential equation we have "guessed" has required more than 3000 terms in the series expansion. Note that a linear differential equation of the minimal order is not, necessarily, the simplest one, as far as the number of terms in the series expansion needed to guess it, is concerned. We have already encountered such a situation in [9].

Recalling $\mathcal{L}_6^{(3)}$, the order six linear differential operator corresponding to $\tilde{\chi}_d^{(3)}(x)$, one finally finds that the linear differential operators $\mathcal{L}_n^{(5)}$ we have obtained‡ for $\tilde{\chi}_d^{(5)}(x)$, have the following factorization:

$$\mathcal{L}_n^{(5)} \;=\; \mathcal{L}_{n-6}^{(5)} \cdot \mathcal{L}_6^{(3)} \qquad (42)$$

implying that $\tilde{\chi}_d^{(3)}(x)$ is actually a solution of $\mathcal{L}_n^{(5)}$, the linear differential operator for $\tilde{\chi}_d^{(5)}(x)$. The "russian-doll" structure shows up again. For the linear differential operator $\mathcal{L}_n^{(5)}$ of smallest order ($n = 19$), $\mathcal{L}_{n-6}^{(5)}$ has a polynomial of apparent singularities of degree 331. For larger orders for $\mathcal{L}_{n-6}^{(5)}$ we get smaller apparent polynomials. In Appendix A, we sketch an order twenty linear differential operator $\mathcal{L}_{n-6}^{(5)}$ modulo the prime 32003, which has *no apparent singularities*, and only the "true" singularities of the linear differential operator $\mathcal{L}_n^{(5)}$ for $\tilde{\chi}_d^{(5)}(x)$:

$$(x+1)\cdot(x-1)\cdot(x^2+x+1)\cdot(x^4+x^3+x^2+x+1)\;\cdot x \;=\; 0.$$

We have not yet been able to get the direct sum decomposition of $\mathcal{L}_n^{(5)}$ in this approach modulo prime.

4. Singularities of $\tilde{\chi}_d^{(n)}(t)$

In [12] we found that the form factors $f^{(n)}(N,t)$ have singularities of the form $\ln^n(1-t)$ at $t \to 1$. These singularities come from the factors $[(1-x_j)(1-tx_j)]^{1/2}$ in the integrands of (7) and (9). These factors are also present in the integrands (11) and (13) for $\tilde{\chi}_d^{(n)}(t)$, and, thus, we expect that, for general values of n, there will be powers of $\ln(1-t)$ present in $\tilde{\chi}_d^{(n)}(t)$. We see, from the previous section, that these logarithmic singularities are first seen in the linear differential equation for $\tilde{\chi}_d^{(3)}(t)$.

However, there are additional singularities in $\tilde{\chi}_d^{(2n)}(t)$ coming from the vanishing of the denominator

$$1 \,-\, t^n\, x_1\, x_2\, \cdots\, x_{2n}, \qquad (43)$$

and, in $\tilde{\chi}_d^{(2n+1)}(t)$, coming from the vanishing of the denominator

$$1 \,-\, t^{n+1/2}\, x_1\, x_2\, \cdots\, x_{2n+1} \qquad (44)$$

‡ The method consists in searching, modulo a given prime, the linear differential operator for $\mathcal{L}_6^{(3)}(\tilde{\chi}_d^{(5)})$.

which are not present in $f^{(n)}(t)$. For $\tilde{\chi}_d^{(2n)}$ the vanishing of (43) occurs for $t^n = 1$ at the endpoints $x_1 = x_2 = \cdots = x_{2n} = 1$, and, for $\tilde{\chi}_d^{(2n+1)}$, the vanishing of (44) occurs for $t^{n+1/2} = 1$ at the endpoints $x_1 = x_2 = \cdots = x_{2n+1} = 1$.

When $t \rightarrow 1$ this additional singularity is the simple pole $(1-t)^{-1}$ for both $\tilde{\chi}_d^{(2n)}(t)$ and $\tilde{\chi}_d^{(2n+1)}(t)$. When t approaches the roots of unity $t^n = 1$ (with $t \neq 1$)

$$t_{l,n} = e^{2\pi i l/n} \qquad \text{with} \qquad l = 1, 2, \cdots, n-1, \tag{45}$$

then $\tilde{\chi}_d^{(2n)}(t)$ has a singularity of the form

$$\kappa_{2n} \cdot (t - t_{l,n})^{2n^2-1} \cdot \ln(t - t_{l,n}). \tag{46}$$

Similarly, when t approaches the roots of unity for $\tilde{\chi}_d^{(2n+1)}$ (with $t \neq 1$), namely $t_0^{n+1/2} = 1$, then $\tilde{\chi}_d^{(2n+1)}(t)$ has a singularity of the form $\kappa_{2n+1} \cdot (t^{1/2} - t_0^{1/2})^{(n+1)^2-1/2}$.

These singularities, on the unit circle in the complex t plane (for $t \neq 1$), are the counterparts for the diagonal susceptibility of the singularities found by Nickel [4, 5] on the unit circle $|s| = 1$, with $s = \sinh 2E/kT$, for the $\tilde{\chi}^{(n)}$ of the isotropic Ising model.

There are *no other* values of t in the complex plane for which the functions $\tilde{\chi}_d^{(n)}(t)$ are singular. This is in distinct contrast with the $\tilde{\chi}^{(n)}(s)$ of the isotropic Ising model which have singularities at many other places on the complex s plane [6]-[9],[11]. In this section we derive the behavior of $\tilde{\chi}_d^{(n)}(t)$ at the root of unity points $t^n = 1$ and $t^{n+1/2} = 1$. Appendix B gives the values of the amplitudes at the singular points lying on the unit circle $|t| = 1$, for $\tilde{\chi}_d^{(3)}$ and $\tilde{\chi}_d^{(4)}$, using the matrix connection method [9].

4.1. The singularity in $\tilde{\chi}_d^{(n)}$ at $t = 1$

To extract the dominant singularity in $\tilde{\chi}_d^{(2n)}$ at $t = 1$ we set $t = 1 - \epsilon$ and $x_k = 1 - \epsilon \cdot y_k$ in (11), and set $\epsilon = 0$ wherever possible. Thus we obtain the result that as $t \rightarrow 1$ ($\epsilon \rightarrow 0$)

$$\tilde{\chi}_d^{(2n)} \sim \frac{1}{1-t} \cdot I_d^{(2n)} \tag{47}$$

with:

$$\begin{aligned} I_d^{(2n)} = & \frac{2}{(n!)^2 (2\pi)^{2n}} \int_0^\infty dy_1 \cdots \int_0^\infty dy_{2n} \cdot \prod_{j=1}^{n} \left(\frac{y_{2j}(1+y_{2j})}{y_{2j-1} \cdot (1+y_{2j-1})} \right)^{1/2} \\ & \times \frac{1}{n + y_1 + \cdots + y_{2n}} \prod_{1 \le j \le n} \prod_{1 \le k \le n} (1 + y_{2j-1} + y_{2k})^{-2} \\ & \times \prod_{1 \le j < k \le n} (y_{2j-1} - y_{2k-1})^2 (y_{2j} - y_{2k})^2. \end{aligned} \tag{48}$$

Similarly, we find the singularity in $\tilde{\chi}_d^{(2n+1)}$ at $t = 1$ reads

$$\tilde{\chi}_d^{(2n+1)} \sim \frac{1}{1-t} \cdot I_d^{(2n+1)} \tag{49}$$

with:

$$I_d^{(2n+1)} = \frac{2}{n!\,(n+1)!\,(2\pi)^{2n+1}} \int_0^\infty dy_1 \cdots \int_0^\infty dy_{2n+1}$$

$$\frac{1}{n+y_1+\cdots+y_{2n}}\prod_{j=1}^{n}[(1+y_{2j})\,y_{2j}]^{1/2}\prod_{j=1}^{n+1}[y_{2j-1}(1+y_{2j-1})]^{-1/2}$$
$$\times\prod_{1\le j\le n+1}\,\prod_{1\le k\le n}(1+y_{2j-1}+y_{2k})^{-2}$$
$$\times\prod_{1\le j<k\le n+1}(y_{2j-1}-y_{2k-1})^2\cdot\prod_{1\le j<k\le n}(y_{2j}-y_{2k})^2. \qquad (50)$$

4.2. Singularities for $\tilde{\chi}_d^{(2n)}(t)$ at $t^n=1$ for $t\neq 1$

When $n=2$ the root of unity singularities $t^n=1$, which is not $t=1$, occurs at $t=-1$ where the analysis of the differential equation given in section (3) shows that there is a singularity of the form $(1+t)^7\cdot\ln(1+t)$, (see Appendix B).

To demonstrate, for general n, that the singularity in $\tilde{\chi}_d^{(2n)}(t)$, at the points $t^n=1$, is given by (46), we set $t=t_{l,n}\cdot(1-\epsilon)$ and $x_k=1-\epsilon\cdot y_k$. Furthermore, because the singularity only occurs in a high derivative of $\tilde{\chi}_d^{(2n)}(t)$, we consider the m^{th} derivative of $\tilde{\chi}_d^{(2n)}(t)$, and will eventually see that m should be chosen to be $2n^2-1$. Then using $t=t_{l,n}\cdot(1-\epsilon)$ and $x_k=1-\epsilon\cdot y_k$, in the m^{th} derivative of (11), and setting $t=t_{l,n}$ and $x_k=1$ wherever possible, we obtain a model integral whose leading singularity, at $t=t_{l,m}$, will be the same as the leading singularity in the m^{th} derivative of $\tilde{\chi}_d^{(2n)}(t)$:

$$I_m^{(2n)}(t) \;=\; \frac{\epsilon^{2n^2-1-m}\,2m!}{(n!)^2\,\pi^{2n}\,(1-t_{l,n})^{n^2}}\cdot\int_0^\infty\cdots\int_0^\infty\prod_{j=1}^{n}dy_j \qquad (51)$$
$$\times\,(\frac{y_{2j-1}}{y_{2j}})^{1/2}\cdot\frac{1}{(n+y_1+y_2+\cdots+y_{2n})^{m+1}}$$
$$\times\prod_{1\le j<k\le n}(y_{2j-1}-y_{2k-1})^2(y_{2j}-y_{2k})^2.$$

This expression is formally independent of ϵ when $m=2n^2-1$. When m has this value the integral diverges logarithmically when all y_k become large, such that all ratios y_j/y_k are of order one. We thus conclude that the singularity in $\tilde{\chi}_d^{(2n)}(t)$, at $t=t_{l,n}$, is given by (46).

4.3. Singularities for $\tilde{\chi}_d^{(2n+1)}(t)$ at $t^{n+1/2}=1$ for $t\neq 1$

For the function $\tilde{\chi}_d^{(3)}(t)$ we used in section (3) the independent variable $x=t^{1/2}$, and found, at the points $x_0^2+x_0+1=0$, that there is a singularity of the form $(x-x_0)^{7/2}$, (see Appendix B). An analysis, completely analogous to the analysis given above for $\tilde{\chi}^{(2n)}(t)$, demonstrates that, for all n, the singularity in $\tilde{\chi}_d^{(2n+1)}(t)$, at $x_0^{2n+1}=t_0^{n+1/2}=1$ (with $t\neq 1$) is given by $\kappa_{2n+1}\cdot(x-x_0)^{(n+1)^2-1/2}$.

5. The direct sum structure

Perhaps the most striking feature of the linear differential operators for $\tilde{\chi}_d^{(3)}(t)$ and $\tilde{\chi}_d^{(4)}(t)$ exhibited in section (3) is their decomposition into a *direct sum*. Such a decomposition has previously been seen for the full susceptibility where the linear

differential operator for $\tilde{\chi}^{(3)}$ is the direct sum [6] of the differential operator for $\tilde{\chi}^{(1)}$ and a second linear differential operator. Similarly, the linear operator for $\tilde{\chi}^{(4)}$ is the direct sum [8] of the linear differential operator for $\tilde{\chi}^{(2)}$ and a second linear differential operator. In the papers [6]-[8] the question is posed of how general is the phenomenon of the direct sum decomposition.

In our previous paper [12] on the form factors $f^{(n)}(N,t)$ of the diagonal Ising correlations we found that the linear differential operators of all form factors for $n \le 9$ have a direct sum decomposition, and that this decomposition is surely valid for all values of n.

An inspection of the direct sum decompositions reveals a great deal of structure which is relevant to $\tilde{\chi}_d^{(n)}(t)$. As an example consider the odd form factors $f^{(2n+1)}(N,t)$ relevant for $T > T_c$. From the direct sum decomposition of the linear differential operators [12] for $f^{(2n+1)}(N,t)$, we find the following form:

$$f^{(3)}(N,t) \;=\; \left(\frac{N}{2}+\frac{1}{6}\right)\cdot f^{(1)}(N,t) \;+g^{(3)}(N,t), \qquad (52)$$

$$f^{(5)}(N,t) \;=\; \frac{1}{120}\cdot(15\,N^2\;+40\,N\;+9)\cdot f^{(1)}(N,t) \\ \qquad +\frac{1}{2}(N+1)\cdot g^{(3)}(N,t)\;+g^{(5)}(N,t) \qquad (53)$$

where $t^{N/2}\cdot g^{(n)}(N,t)$ is a homogeneous polynomial of degree n in the complete elliptic integrals K and E, with coefficients which are polynomials in t. If we then sum the decomposition (52) of $f^{(3)}(N,t)$, we find that

$$\tilde{\chi}_d^{(3)}(t) \;=\; \tilde{\chi}_{d,1}^{(3)}(t)\;+\tilde{\chi}_{d,2}^{(3)}(t)\;+\tilde{\chi}_{d,3}^{(3)} \qquad (54)$$

where¶

$$\tilde{\chi}_{d,1}^{(3)}(t) \;=\; \frac{1}{6}\sum_{N=-\infty}^{\infty} f^{(1)}(N,t) \;=\; \frac{1}{6}\,\tilde{\chi}_d^{(1)}, \qquad (55)$$

$$\tilde{\chi}_{d,2}^{(3)} \;=\; \sum_{N=-\infty}^{\infty}\frac{N}{2}\cdot f^{(1)}(N,t) \qquad (56)$$

where $\tilde{\chi}_{d,2}^{(3)}$ is the solution of the linear differential operator $L_2^{(3)}$ (which is regular at $x = t^{1/2} = 0$) and where

$$\tilde{\chi}_{d,3}^{(3)} \;=\; \sum_{N=-\infty}^{\infty} g^{(3)}(N,t) \;=\; b_1\cdot\tilde{\chi}_d^{(1)}(t)\;+sol(L_3^{(3)}) \qquad (57)$$

with♯ $b_1 \ne -1/6$. Thus we see that the direct sum decomposition of the linear differential operators for $\tilde{\chi}_d^{(3)}(t)$ follows immediately from the direct sum decomposition of the differential operators for $f^{(3)}(N,t)$.

For $f^{(5)}(N,t)$, the sum (53) can be written in terms of $f^{(3)}(N,t)$ as

$$f^{(5)}(N,t) \;=\; -\frac{1}{120}\left(15\,N^2\;+1\right)\cdot f^{(1)}(N,t) \\ \qquad +\frac{1}{2}\,(N+1)\cdot f^{(3)}(N,t)\;+g^{(5)}(N,t)$$

¶ The integer N should be understood as $|N|$ in all the summations below.

♯ In fact, and since the full factorization of $\mathcal{L}_6^{(3)}$ is known, the projection of $\tilde{\chi}_d^{(1)}(t)$ in $\tilde{\chi}_d^{(3)}(t)$ can be computed and one finds $b_1 = 1/6$.

Summing on N, one obtains:

$$\chi_d^{(5)} \;=\; -\frac{1}{120} \sum_N f^{(1)}(N,t) \,-\, \frac{1}{8} \sum_N N^2 \cdot f^{(1)}(N,t) \,+\, \frac{1}{2} \sum_N f^{(3)}(N,t) \\ +\, \frac{1}{2} \sum_N N \cdot f^{(3)}(N,t) \,+\, \sum_N g^{(5)}(N,t). \qquad (58)$$

The first term, at the right-hand-side of (58), satisfies the linear differential equation of order-one corresponding to $\chi_d^{(1)}$, (i.e. $L_1^{(3)}$ in (28)). The second term satisfies an order-one linear differential equation. The third term is just $\chi_d^{(3)}$, up to a constant, a result that we knew from the modulo prime method of section (3.3). We have found the differential operator corresponding to the fourth term which is of order eight and has, in its direct sum, the linear differential operator $L_2^{(3)}$ given in (28). As was the case for the sum in (57), one might imagine that the linear differential operator corresponding to the last term in (58) will contain some of the differential operators related to the previous terms.

We consider now the $\tilde{\chi}_d^{(2n)}(t)$. In [12] we found that there is a direct sum decomposition for the linear differential operator for $f^{(2n)}(N,t)$, just as there was one for $f^{(2n+1)}(N,t)$. From these direct sum decompositions, we find:

$$f^{(2)}(N,t) \;=\; \frac{N}{2} \,+ g^{(2)}(N,t), \qquad (59)$$

$$f^{(4)}(N,t) \;=\; \frac{N\cdot(N+2)}{8} \,+ \left(\frac{N}{2} + \frac{1}{3}\right) \cdot g^{(2)}(N,t) \,+ g^{(4)}(N,t) \\ =\; \frac{N\cdot(2-3N)}{24} \,+ \left(\frac{N}{2} + \frac{1}{3}\right) \cdot f^{(2)}(N,t) + g^{(4)}(N,t) \qquad (60)$$

Similarly‡, the direct sum decomposition of the linear differential operators for $f^{(2n)}(N,t)$ will lead to the direct sum decomposition of the differential operators for $\tilde{\chi}_d^{(2n)}(t)$ for all n.

We conclude that there will be a direct sum decomposition for the linear differential operators for $\tilde{\chi}_d^{(n)}(t)$, *for all* n, and that this direct sum decomposition is inherited from the direct sum decomposition for the linear differential operators of the form factors of the Ising model.

Remark: It is worth noting that we have also performed a large set of calculations (that will not be detailed here) on more "artificial" toy susceptibilities like :

$$\chi_{toy} \;=\; \sum_{j=1}^{\infty} \sum_{N=1}^{\infty} N^2 \cdot f^{(j)}(N,t). \qquad (61)$$

Similarly, for the corresponding j-particle contributions $\chi_{toy}^{(j)}$ we found for the first values of j ($j = 1, 2, 3, 4$), the corresponding Fuchsian linear differential operators. The singularities of these Fuchsian linear differential equations and of the corresponding j-fold integrals, are totally and utterly similar to the one's of the diagonal susceptibility analyzed in this paper. Again we have equations totally similar to (54), (55), (56). This confirms, very clearly, that the direct sum decomposition for the j-particle contributions $\chi_{toy}^{(j)}$ is straightforwardly inherited from the direct sum decomposition for the form factors of the model.

‡ Note that the first terms in (59) or (60) do not lead to divergent sums. They are balanced by the sums on the last terms, i.e., $g^{(2)}(N,t)$ and $g^{(4)}(N,t)$.

5.1. Resummations

There is one further feature of these decompositions which must be mentioned. Namely, the sums of the form

$$\sum_{N=-\infty}^{\infty} N^p \cdot f^{(2n+1)}(N,t) \qquad (62)$$

will diverge at $t = 1$ as $(1-t^{1/2})^{-p-1}$. Thus, for example, the solution of $L_2^{(3)}$, given in (32), diverges at $t^{1/2} = 1$ as $(1-t^{1/2})^{-2}$. We see, from (34), that the leading singularity in the solution to $L_3^{(3)}$ will also diverge, at $t^{1/2} = 1$, as $(1-t^{1/2})^{-2}$. However, the full solution for $\tilde{\chi}_d^{(3)}(t)$ must diverge, at $t \rightarrow 1$, only as $(1-t^{1/2})^{-1}$, and, therefore, there must be cancellations between terms in the direct sum. This phenomenon will happen for all $\tilde{\chi}_d^{(2n+1)}(t)$ where the cancellations become more extensive as n increases. For heuristic reasons let us consider $\chi_d^{(1)}$ versus $f^{(1)}(N)$.

The diagonal susceptibility of order one is defined as

$$\chi_d^{(1)} \;=\; \sum_{N=-\infty}^{\infty} f^{(1)}(N,t) \;=\; f^{(1)}(0,t) \;+2 \sum_{N=1}^{\infty} f^{(1)}(N,t) \qquad (63)$$

with

$$f^{(1)}(N,t) \;=\; \frac{(1/2)_N}{N!} \cdot t^{N/2} \cdot {}_2F_1\left(1/2, 1/2+N; 1+N; t\right). \qquad (64)$$

Writing in the sum over N, the hypergeometric function as a series, one obtains:

$$\chi_d^{(1)} \;=\; K(t) \;+2 \sum_{k=0}^{\infty}\sum_{N=1}^{\infty} \frac{(1/2)_k\,(1/2)_N\,(1/2+N)_k}{(1+N)_k k! N!} \cdot t^{k+N/2} \qquad (65)$$

with $(x)_k$ denoting the Pochhammer symbol, and $K(t)$ is as defined in (33). One may shift N, separate the even from the odd N, and try to sum. One may also generate the expansion, find the linear ODE and solve. Remarkably the sum in (63) reduces to a simple expression in terms of the complete elliptic integral of the first kind:

$$2\sum_{N=1}^{\infty} f^{(1)}(N,t) \;=\; -\frac{\sqrt{t}}{t-1} - \frac{1}{t-1} \;-K(t). \qquad (66)$$

This remarkable identity (66) explains why a sum like (63), where each term is polynomial expression of the complete elliptic integral of the first kind and of the second kind, succeeds to reduce to a simple rational expression in $t^{1/2}$:

$$\chi_d^{(1)} \;=\; K(t) - \frac{\sqrt{t}}{t-1} \;-\frac{1}{t-1} \;-K(t) \;=\; \frac{1}{1-\sqrt{t}}. \qquad (67)$$

6. The singularities in the diagonal susceptibility

Thus far we have discussed the n^{th} particle form factor contribution of $\tilde{\chi}_d^{(n)}(t)$. It remains to use this information to study the diagonal susceptibility $\chi_d(t)$ itself, and, for this, we need to consider several problems which also occur for the computation of the full susceptibility $\chi(s)$.

We computed the susceptibilities $\chi_{d\pm}$ by summing the form factor expansions (6) and (8) of the diagonal correlation functions $C(N,N)$ over all integer values of N. In doing this we have interchanged the sum over position N with the sum over form

factors n. In field theory language we have interchanged the high energy limit with the sum over n particle intermediate states. This interchange is universally done in both statistical mechanics and in field theory, but should, in principle, be justified.

Let us assume that this interchange of the sum over N and n can be made. Then the behavior of χ_d, as $T \to T_c$, can be studied from the behavior of $\tilde{\chi}^{(n)}(t)$ as $t \to 1$ if we make the additional assumption that the limit $T \to T_c$ can also be interchanged with the sum over n. We then may use (47) in (10) to find, as $T \to T_c-$, that

$$kT \cdot \chi_{d-} \sim (1-t)^{-3/4} \cdot \sum_{n=1}^{\infty} I_d^{(2n)} \qquad (68)$$

and, similarly, by using (49) in (12) we find, as $T \to T_c+$, that:

$$kT \cdot \chi_{d+} \sim (1-t)^{-3/4} \cdot \sum_{n=0}^{\infty} I_d^{(2n+1)}. \qquad (69)$$

The sums in (68) and (69) must be shown to converge if these estimates of the critical behavior are to be correct. For the full susceptibility, similar convergence has been recently demonstrated by Bailey, Borwein and Crandall [16].

The single pole divergence, which occurs in $\tilde{\chi}_d^{(n)}(t)$ for each n, is the analogue, for the diagonal susceptibility, of the double pole divergence $(1-s)^2$ in the $\tilde{\chi}^{(n)}(s)$ of the full susceptibility. In both cases this divergence occurs from terms in the integrand of $\tilde{\chi}_d^{(n)}(t)$ (or $\tilde{\chi}^{(n)}(s)$) which are not present in the corresponding integrals for the form factor representation of the correlation function. These divergences may be said to come from long distance effects, and are captured in the scaling theory of the correlation functions.

There are further singularities in $\tilde{\chi}_d^{(n)}(t)$ which come from the pinch singularities [11] of the square root branch points $[(1-x)(1-tx)]^{1/2}$ in the integrands which are also present in the form factors $f^{(n)}(N,t)$. In the form factors $f^{(n)}(N,t)$, singularities give divergent terms $\ln^n(1-t)$ which, term by term, would give the dominant contribution $(1-t)^{1/4} \cdot \sum_n \ln^n(1-t)\, S_n(1-t)$ to the correlation function. However, the overall factor $(1-t)^{1/4}$ is absent in the diagonal correlation $C(N,N)$ and is thus cancelled by the infinite sum on n. We thus regain the original expansion of $C(N,N)$ as an $N \times N$ determinant whose singularities, at $t = 1$, are of the form:

$$(1-t)^{N^2} \cdot \ln^N(1-t). \qquad (70)$$

These singularities in $C(N,N)$, which come from the summation of the form factors over n, may be said to be short distance singularities.

In $\tilde{\chi}_d^{(n)}(t)$, for $n = 1, 2$, the results of section (2) show that there are no logarithmic terms. However, for $n = 3, 4$, logarithmic terms occur and we presume (but have not yet demonstrated) for arbitrary $n > 4$, that $\tilde{\chi}_d^{(n)}(t)$ will have logarithmic terms. However, just as was the case for $f^{(n)}(N,t)$, these powers of log's must be summed over all n, and, again, just as for $f^{(n)}(N,t)$, this sum must cancel the factor of $(1-t)^{1/4}$. It will thus give terms of the form (70) for $\chi_d(t)$ which are the counterpart of the similar terms in the full susceptibility $\chi(s)$, and have been studied in detail by Orrick, Nickel, Guttmann and Perk [14].

It remains to discuss the singularities of the diagonal susceptibility on the unit circle $|t| = 1$. These singularities not only have the property that they become dense on the unit circle as $n \to \infty$, but since they are roots of unity, they accumulate according to a uniform distribution, in contrast with the density of the "nickellian" singularities

(see (3.24) in [14] and (1) in [11]), which coincides with the density of zeroes of the partition function without magnetic field given in [17]. Another qualitative difference between the diagonal susceptibility, and the full susceptibility, is that these unit circle singularities for the diagonal susceptibility are logarithmic for $T < T_c$, and are square root type for $T > T_c$, whereas, for the full susceptibility the singularities for $T > T_c$ are logarithmic, and the singularities, for $T < T_c$, are of square root type. In all cases the amplitudes of the singularities depend strongly on n which would seem to prevent any possible cancellation between singularities in sets such as $\tilde{\chi}_d^{(2mn)}(t)$, with n fixed and $m = 1, 2, \cdots$, which have the locations of some of the singularities at coinciding positions. Therefore the arguments used by Nickel [4]-[5] to conjecture a natural boundary in the full Ising susceptibility will similarly suggest that there is a natural boundary in the diagonal Ising susceptibility as well. As is the case with the full susceptibility a more rigorous argument would be most desirable.

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Appendix A. Linear differential operators

Recalling the factorization relation (42) of the linear differential operator $\mathcal{L}_n^{(5)}$ corresponding to $\tilde{\chi}_d^{(5)}(x)$ into a linear differential operator $\mathcal{L}_{n-6}^{(5)}$ and the linear differential operator corresponding to $\tilde{\chi}_d^{(3)}(x)$:

$$\mathcal{L}_n^{(5)} \; = \; \mathcal{L}_{n-6}^{(5)} \cdot \mathcal{L}_6^{(3)}, \qquad \text{(A.1)}$$

the smallest order for the linear differential operator $\mathcal{L}_{n-6}^{(5)}$ is thirteen, but this yields a large set of apparent singularities. For an order twenty, the linear differential operator $\mathcal{L}_{20}^{(5)}$ has no apparent singularities, and requires less terms, in the series expansion, to be guessed. Let us sketch this order twenty linear differential operator $\mathcal{L}_{20}^{(5)}$ modulo the prime 32003:

$$\mathcal{L}_{20}^{(5)} \; = \; \sum_{i=0}^{i=20} Q_i \cdot Dx^i, \qquad \text{with} \qquad \text{(A.2)}$$

$$\begin{aligned} Q_{20} \; &= \; (1+x)^6 \left(x^2-1\right)^3 \left(x^3-1\right)^2 \, \left(x^5-1\right) \, x^{10} \qquad \text{(A.3)} \\ &= \; (1+x)^9 \, (x-1)^6 \, (1+x+x^2)^2 \, (1+x+x^2+x^3+x^4) \cdot x^{10} \\ Q_m \; &= \; (1+x)^{\alpha(9,\, m)} \cdot (x-1)^{\alpha(6,\, m)} \cdot \, (1+x+x^2)^{\alpha(2,\, m)} \cdot x^{\alpha(10,\, m)} \cdot q_m \\ \text{where} \qquad & m \; = \; 0, \, 1, \, 2, \, \cdots, \, 19 \qquad \text{and} \\ & \alpha(N, \, m) \; = \; sup(0, \, N-20+m) \end{aligned}$$

and where the polynomials‡ q_m read respectively :

$$q_{19} \; = \; 29006 \cdot (x^2 + 1466\, x \, + 22107) \cdot P_{19}^{(17)} \cdot R_{19}^{(17)} \cdot P_{19}^{(13)} \cdot P_{19}^{(6)},$$

‡ The polynomials q_m are modulo the prime 32003.

$$\begin{aligned}
q_{18} \; &= \; 23383 \cdot (x+13780) \cdot (x+13647) \cdot P_{18}^{(34)} \cdot P_{18}^{(10)} \cdot P_{18}^{(8)} \cdot P_{18}^{(5)}, \\
q_{17} \; &= \; 23894 \cdot (x^2+13626\,x\,+7861) \cdot P_{17}^{(59)}, \\
q_{16} \; &= \; 31481 \cdot (x+31392) \cdot P_{16}^{(19)} \cdot P_{16}^{(18)} \cdot P_{16}^{(14)} \cdot P_{16}^{(11)}, \\
q_{15} \; &= \; 13122 \cdot (x+2555) \cdot P_{15}^{(40)} \cdot P_{15}^{(21)} \cdot P_{15}^{(3)}, \\
q_{14} \; &= \; 1689 \cdot P_{14}^{(57)} \cdot P_{14}^{(10)}, \\
q_{13} \; &= \; 851 \cdot (x+6919) \cdot P_{13}^{(67)}, \\
q_{12} \; &= \; 542 \cdot P_{12}^{(25)} \cdot P_{12}^{(21)} \cdot P_{12}^{(11)} \cdot P_{12}^{(5)} \cdot P_{12}^{(4)} \cdot P_{12}^{(3)}, \\
q_{11} \; &= \; 26141 \cdot (x+1) \cdot P_{11}^{(58)} \cdot P_{11}^{(5)} \cdot P_{11}^{(4)} \cdot P_{11}^{(2)}, \\
q_{10} \; &= \; 31757 \cdot x \cdot (x+14054) \cdot P_{10}^{(55)} \cdot P_{10}^{(11)} \cdot P_{10}^{(2)}, \\
q_{9} \; &= \; 31477 \cdot P_{9}^{(45)} \cdot P_{9}^{(15)} \cdot P_{9}^{(5)} \cdot P_{9}^{(2)} \cdot R_{9}^{(2)}, \\
q_{8} \; &= \; 28150 \cdot P_{8}^{(62)} \cdot P_{8}^{(4)} \cdot P_{8}^{(2)}, \\
q_{7} \; &= \; 2111 \cdot (x+16608) \cdot P_{7}^{(64)} \cdot P_{7}^{(2)}, \\
q_{6} \; &= \; 21300 \cdot (x+15054) \cdot (x+20971) \cdot P_{6}^{(37)} \cdot P_{6}^{(23)} \cdot P_{6}^{(2)} \cdot R_{6}^{(2)}, \\
q_{5} \; &= \; 8699 \cdot (x+1134) \cdot P_{5}^{(57)} \cdot P_{5}^{(4)} \cdot P_{5}^{(3)}, \\
q_{4} \; &= \; 7621 \cdot (x+27997) \cdot P_{4}^{(7)} \cdot P_{4}^{(42)} \cdot P_{4}^{(8)} \cdot P_{4}^{(6)}, \\
q_{3} \; &= \; 18283 \cdot (x+26460) \cdot P_{3}^{(58)} \cdot P_{3}^{(4)}, \\
q_{2} \; &= \; 2235 \cdot (x+8688) \cdot (x+20285) \\
&\quad \times (x+24023) \cdot P_{2}^{(34)} \cdot P_{2}^{(12)} \cdot P_{2}^{(8)} \cdot P_{2}^{(3)} \cdot P_{2}^{(2)}, \\
q_{1} \; &= \; 2139 \cdot (x+19284) \cdot (x+19339) \cdot P_{1}^{(55)} \cdot P_{1}^{(4)}, \\
q_{0} \; &= \; 23255 \cdot (x+30075) \cdot (x+13139) \cdot P_{0}^{(41)} \cdot P_{0}^{(15)} \cdot P_{0}^{(2)},
\end{aligned}$$

where the polynomials $P_m^{(N)}$ or $R_m^{(N)}$ are polynomials of degree N :

$$P_m^{(N)} \; = \; x^N \, + \, \cdots \qquad \text{(A.4)}$$

Appendix B. Singular behavior of $\tilde{\chi}_d^{(3)}$ and $\tilde{\chi}_d^{(4)}$

Using the matrix connection method [9], we have obtained the values of the amplitudes at the singularities, lying on the unit circle $|t| = 1$, of $\tilde{\chi}_d^{(3)}$ and $\tilde{\chi}_d^{(4)}$.

The linear differential equation for $\chi_d^{(3)}$ has the singularities $x = 0, 1, -1, \infty$ and the roots of $1+x+x^2$ $(t = x^2)$. The singular behavior of $\chi_d^{(3)}$ at the singular point x_s, denoted by $\chi_d^{(3)}$ (singular, x_s) read (in the local variable $u = x - x_s$):

$$\tilde{\chi}_d^{(3)}\,(\mathrm{singular}, 1) \; = \; \Big({{1} \over {3}} \, + {{3} \over {4\pi}} \, + {{a} \over {6}} \Big) \cdot {{1} \over {u}} \, \, + {{1} \over {8\,\pi}} \cdot \ln(u) \qquad \text{(B.1)}$$

with $a = 0.469629259 \cdots$

$$\tilde{\chi}_d^{(3)}\,(\mathrm{singular}, -1) \; = \; {{1} \over {4\pi^2}} \, \ln(u)^2 \, + \Big({{1} \over {4\pi}} \, - {{2\,\ln(2)\,-1} \over {2\,\pi^2}} \Big) \cdot \ln(u),$$

$$\tilde{\chi}_d^{(3)}\,(\mathrm{singular}, x_0) \; = \; -{{1} \over {6}} \, (1-i) \, (1+i\,\sqrt{3}) \cdot b \cdot u^{7/2}$$

where $x_0 = -1/2 + i\sqrt{3}/2$ and with $b = 0.203122784 \cdots$

The linear differential equation for $\tilde{\chi}_d^{(4)}$ has the singularities $t = 0, 1, -1, \infty$. Denoting by $u = t - t_s$ the local expansion variable, where t_s is the singularity, the singular behavior for $\tilde{\chi}_d^{(4)}$ read

$$\tilde{\chi}_d^{(4)}\,(\mathrm{singular}, 1) \; = \; \Bigl({{1} \over {\pi^2}} \; - {{1} \over {4}} \; - {{c} \over {8}} \Bigr) \cdot {{1} \over {u}} \; - {{1} \over {8\,\pi^2}} \cdot \ln(u)^2 \; + {{8\,\ln(2) \, - 7 \, - i\,2\,\pi} \over {8\,\pi^2}} \cdot \ln(u)$$

with $c = -1.120950429 \cdots$ and:

$$\tilde{\chi}_d^{(4)}\,(\mathrm{singular}, -1) \; = \; {{1} \over {13440\,\pi^2}} \cdot u^7 \cdot \ln(u).$$

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